* The temporal Heisenberg inequality

- Ehrenfest theorem:

$$\frac{d}{dt}\langle A\rangle_\varphi = \frac{1}{i\hbar}\langle [A,H]\rangle_\varphi \qquad \Big\|\ \langle\cdot\rangle_\varphi = \langle\varphi(t)|\cdot|\varphi(t)\rangle$$

- uncertainty relation: $\langle(\Delta A)^2\rangle_\varphi \langle(\Delta B)^2\rangle_\varphi \geq \frac{1}{4}|\langle[A,B]\rangle_\varphi|^2$
  $\langle(\Delta A)^2\rangle_\varphi \equiv (\Delta_\varphi A)^2$.

  Let's put H into B!

$$\to \quad \Delta_\varphi H\, \Delta_\varphi A \geq \frac{1}{2}|\langle[A,H]\rangle_\varphi| = \frac{1}{2}\hbar\left|\frac{d}{dt}\langle A\rangle_\varphi\right|$$

If we define the time $\tau_\varphi(A)$ as

$$\frac{1}{\tau_\varphi(A)} \equiv \left|\frac{d\langle A\rangle_\varphi}{dt}\right|\frac{1}{\Delta_\varphi A},$$

then τ_φ = characteristic time for the expectation value of A to change by $\Delta_\varphi A$.

$$\Rightarrow \quad \Delta_\varphi H\, \tau_\varphi(A) \geq \frac{1}{2}\hbar \quad \Rightarrow \quad \Delta E\, \Delta t \gtrsim \frac{1}{2}\hbar \quad //$$

Energy spread — characteristic evolution time.

2.3 Simple Harmonic oscillator

(1) Energy eigenkets. (Dirac's operator method)

$$H = \frac{\tilde{p}^2}{2m} + \frac{1}{2}m\omega^2\tilde{x}^2 = \hbar\omega\left(\tilde{a}^+\tilde{a} + \frac{1}{2}\right)$$
$$\equiv \hbar\omega\left(\tilde{N} + \frac{1}{2}\right)$$

def. annihilation operator / creation operator

$$\begin{cases} \tilde{a} = \sqrt{\dfrac{m\omega}{2\hbar}}\left(\tilde{x} + i\dfrac{\tilde{p}}{m\omega}\right) \\ \tilde{a}^+ = \sqrt{\dfrac{m\omega}{2\hbar}}\left(\tilde{x} - i\dfrac{\tilde{p}}{m\omega}\right) \end{cases} \Rightarrow \quad \begin{cases} \tilde{x} = \dfrac{x_0}{\sqrt{2}}(\tilde{a} + \tilde{a}^+) \\ \tilde{p} = i\dfrac{\hbar}{\sqrt{2}x_0}(-\tilde{a} + \tilde{a}^+) \end{cases}$$

$$\tilde{N} = \tilde{a}^+\tilde{a} \qquad \Big\|\ x_0 = \sqrt{\frac{\hbar}{m\omega}}$$

$\to$ Commutation relation $[\tilde{a}, \tilde{a}^+] = 1$

$\Rightarrow [H, \tilde{N}] = 0$; There's simultaneous eigenkets

$\therefore$ $\underline{H|n\rangle = (n+\frac{1}{2})\hbar\omega|n\rangle}$ $\quad\|\quad$ $\tilde{N}|n\rangle = n|n\rangle$

same eigenkets $\{|n\rangle\}$.

$\Rightarrow$ $\boxed{E_n = (n+\frac{1}{2})\hbar\omega}$ $\quad\|\quad$ $H|n\rangle = E_n|n\rangle$

Q. What's "n", then? We know that $n = 0, 1, 2, \cdots$

But, how do we know that from what we have?

Let's look at the commutation relation:

$$\underline{[\tilde{N}, \tilde{a}]} = [\tilde{a}^\dagger\tilde{a}, \tilde{a}] = \tilde{a}^\dagger\underbrace{[\tilde{a}, \tilde{a}]}_{=0} + \underbrace{[\tilde{a}^\dagger, \tilde{a}]}_{=-1}\tilde{a}$$

$$\underline{= -\tilde{a}}$$

likewise, $\underline{[\tilde{N}, \tilde{a}^\dagger] = \tilde{a}^\dagger}$.

Now try $\underline{\tilde{N}\tilde{a}^\dagger|n\rangle} = ([N, \tilde{a}^\dagger] + \tilde{a}^\dagger\tilde{N})|n\rangle$

switch!

$$= \tilde{a}^\dagger(\tilde{N}+1)|n\rangle$$

$$\Rightarrow \underline{\tilde{N}\tilde{a}^\dagger|n\rangle = (n+1)\tilde{a}^\dagger|n\rangle}$$

Thus, $\tilde{a}^\dagger$ is a creation operator as it makes $n \to n+1$

likewise, $\underline{\tilde{N}\tilde{a}|n\rangle} = ([\tilde{N}, \tilde{a}] + \tilde{a}\tilde{N})|n\rangle$

$$\underline{= (n-1)|n\rangle}$$

$\Rightarrow$ $\tilde{a}$ is an annihilation operator.

$(n \to n-1)$

$$\left[\begin{array}{l}\tilde{N}\tilde{a}^{+}|n\rangle = (n+1)\tilde{a}^{+}|n\rangle \\ \tilde{N}\tilde{a}|n\rangle = (n-1)\tilde{a}|n\rangle\end{array}\right. \xrightarrow{\text{implies}} \boxed{\begin{array}{l}\tilde{a}^{+}|n\rangle = c_{+}|n+1\rangle \\ \tilde{a}|n\rangle = c_{-}|n-1\rangle\end{array}}$$

* Let's check : $\tilde{N}\tilde{a}^{+}|n\rangle = (n+1)\tilde{a}^{+}|n\rangle$ || $C_{\pm}$: c-number.

$$\Rightarrow \tilde{N}\cdot \cancel{c_{+}}|n+1\rangle = (n+1)\cdot\cancel{c_{+}}\cdot|n+1\rangle$$

$$\tilde{N}|n+1\rangle = (n+1)|n+1\rangle \quad ; \text{ OK.}$$

Now, Let's determine $C_{\pm}$.

To recover $\tilde{N}|n\rangle = n|n\rangle$,

$$\langle n|\tilde{a}^{+}\tilde{a}|n\rangle = |c_{-}|^{2}\langle n-1|n-1\rangle = n$$

$\therefore$ $\underline{c_{-} = \sqrt{n}}$ // || choose $C_{\pm}$ to be real and positive.

likewise, $\langle n|\tilde{a}^{+}\tilde{a}|n\rangle = \langle n|\cdot\left(\tilde{a}\tilde{a}^{+} - \underbrace{\tilde{a}\tilde{a}^{+} + \tilde{a}^{+}\tilde{a}}_{=[\tilde{a}^{+},\tilde{a}]=-1}\right)\cdot|n\rangle$

$$= \langle n|\tilde{a}\tilde{a}^{+}|n\rangle - 1$$

$\Rightarrow$ $|C_{+}|^{2} = n+1$ $\therefore$ $\underline{C_{+} = \sqrt{n+1}}$ //

Now, we know that, if $(n, |n\rangle)$ is the eigenpair,

So ARE. $\{\cdots (n-2, |n-2\rangle), (n-1, |n-1\rangle), (n+1, |n+1\rangle), (n+2, |n+2\rangle), \cdots\}$.

; Not enough to determine n.
Is there any lower bound ?

$\boxed{\therefore\ n = 0, 1, 2, 3, 4, \cdots}$

$$\Rightarrow \langle n|\tilde{N}|n\rangle = \left(\langle n|\tilde{a}^{+}\right)\cdot\left(a|n\rangle\right) \geq 0.$$

$$\therefore \underline{n \geq 0}$$

⇒ Ground-state energy , Eigen state

$$E_0 = \frac{1}{2}\hbar\omega \quad \leftarrow \quad |0\rangle \qquad \| \qquad |n+1\rangle = \frac{\tilde{a}^+}{\sqrt{n+1}}|n\rangle$$

$\hookrightarrow |1\rangle$

$\hookrightarrow |2\rangle$

$\hookrightarrow \vdots$

⇒ n-th excited state

$$E_n = \left(n+\frac{1}{2}\right)\hbar\omega \quad \leftarrow \quad \left[\frac{(\tilde{a}^+)^n}{\sqrt{n!}}\right]|0\rangle \equiv |n\rangle .$$

$n = 0, 1, 2, \cdots$

* matrix representation in the basis of $\{|n\rangle\}$.

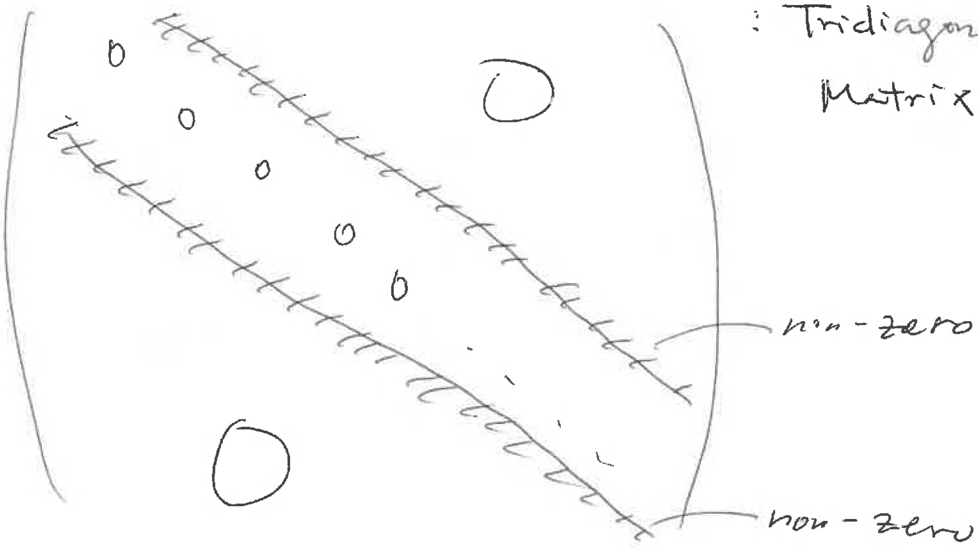
- $\langle n'|\tilde{a}|n\rangle = \sqrt{n}\,\delta_{n',n-1}$, $\langle n'|\tilde{a}^+|n\rangle = \sqrt{n+1}\,\delta_{n',n+1}$

- $\tilde{x}$ and $\tilde{p}$?

$$\begin{cases} \tilde{x} = \sqrt{\dfrac{\hbar}{2m\omega}}\,(\tilde{a}+\tilde{a}^+) \\ \tilde{p} = i\sqrt{\dfrac{m\hbar\omega}{2}}\,(-\tilde{a}+\tilde{a}^+) \end{cases}$$

$$\Rightarrow \langle n'|\tilde{x}|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n}\,\delta_{n',n-1} + \sqrt{n+1}\,\delta_{n',n+1}\right),$$

$$\langle n'|\tilde{p}|n\rangle = i\sqrt{\frac{m\hbar\omega}{2}}\left(-\sqrt{n}\,\delta_{n',n-1} + \sqrt{n+1}\,\delta_{n',n+1}\right).$$

⇒ 

: Tridiagonal Matrix.

- Energy eigenfunction in x-space.

- <u>ground-state wave function</u> $\langle x|0\rangle$:

Ground-state ket $|0\rangle$ statisfies

$$\tilde{a}|0\rangle = 0 \implies \langle x|\tilde{a}|0\rangle = \sqrt{\frac{m\omega}{2\hbar}}\langle x|\,\tilde{x} + i\frac{\tilde{p}}{m\omega}|0\rangle$$

$$\implies \left(x + x_0^2\frac{\partial}{\partial x}\right)\langle x|0\rangle = 0 \qquad \Big\| \quad x_0^2 = \frac{\hbar}{m\omega}$$

1st. order diff. eq.

$$\langle x|0\rangle = A\, e^{-\frac{1}{2}\frac{x^2}{x_0^2}} \qquad \oplus \text{ normalization } \int_{-\infty}^{\infty} dx\, \langle x|0\rangle = 1$$

$$= \frac{1}{\pi^{1/4}\sqrt{x_0}}\exp\left[-\frac{1}{2}\left(\frac{x}{x_0}\right)^2\right].$$

- excited states.

$$\langle x|1\rangle = \langle x|\tilde{a}^{+}|0\rangle = \sqrt{\frac{m\omega}{2\hbar}}\langle x|\,\tilde{x} - i\frac{\tilde{p}}{m\omega}|0\rangle$$

$$= \frac{1}{\sqrt{2}x_0}\left(x - x_0^2\frac{\partial}{\partial x}\right)\langle x|0\rangle$$

$\vdots$

$$\langle x|n\rangle = \frac{1}{\sqrt{n!}}\langle x|(\tilde{a}^{+})^n|0\rangle = \frac{1}{\sqrt{n!}}\left(\frac{1}{\sqrt{2}x_0}\right)^n\left(x - x_0^2\frac{\partial}{\partial x}\right)^n\langle x|0\rangle.$$

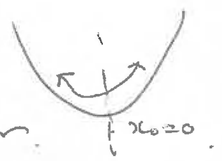
$$= \frac{1}{\pi^{1/4}\sqrt{2^n n!}}\left(\frac{1}{x_0^{\,n+\frac{1}{2}}}\right)\left(x - x_0^2\frac{\partial}{\partial x}\right)^n\exp\left(-\frac{1}{2}\left(\frac{x}{x_0}\right)^2\right).$$

- Let's check up what we know from the uncertainty principle. <u>$\Delta x\,\Delta p \gtrsim \frac{\hbar}{2}$</u>

- classical Hamiltonian :

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = E \quad \text{(Energy)}.$$

we know $\Delta p = p$, $\Delta x = x$ since its an oscillator.



$$E = \frac{\Delta p^2}{2m} + \frac{1}{2}m\omega^2 \Delta x^2 \geq \underbrace{\frac{\hbar^2}{8m}\Delta x^{-2} + \frac{1}{2}m\omega^2\Delta x^2}$$

has minimum at $\Delta x^2 = \frac{\hbar}{2m\omega}$

$(E_{min} = \frac{\hbar}{2}\omega)$

$\Delta p^2 = \frac{\hbar m\omega}{2}$

$$\geq \frac{1}{4}\hbar\omega + \frac{1}{4}\hbar\omega$$

$$\geq \frac{1}{2}\hbar\omega = E_{ground}$$

- Let's verify these steps with $\langle x|0\rangle$.

$$\langle \tilde{x} \rangle = 0 , \quad \langle \tilde{p} \rangle = 0 .$$

$$\langle \tilde{x}^2 \rangle = \left\langle \frac{\hbar}{2m\omega} (\tilde{a} + \tilde{a}^+)^2 \right\rangle = \frac{\hbar}{2m\omega} \langle 0| (\tilde{a}^2 + \tilde{a}^{+2} + \tilde{a}^+\tilde{a} + \underbrace{\tilde{a}\tilde{a}^+}_{1 + \tilde{a}^+\tilde{a}}) |0\rangle$$

$$= \frac{\hbar}{2m\omega}$$

Similarly, $\langle \tilde{p}^2 \rangle = \frac{\hbar m \omega}{2}$ $\Leftrightarrow$ $\Delta x^2 = \frac{\hbar}{2m\omega}$, $\Delta p^2 = \frac{\hbar m\omega}{2}$.

$$\left\langle \frac{\tilde{p}^2}{2m} \right\rangle = \frac{1}{4}\hbar\omega , \quad \left\langle \frac{1}{2}m\omega^2\tilde{x}^2 \right\rangle = \frac{1}{4}\hbar\omega$$

$$E = \langle H \rangle = \frac{1}{2}\hbar\omega .$$

$$\langle (\Delta\tilde{x})^2 \rangle \langle (\Delta\tilde{p})^2 \rangle = \frac{\hbar}{2m\omega} \cdot \frac{\hbar m\omega}{2} = \frac{\hbar^2}{4}$$

for the nth state,

$$\langle (\Delta\tilde{x})^2 \rangle \langle (\Delta\tilde{p})^2 \rangle = \left(n + \frac{1}{2}\right)^2 \hbar^2 \geq \frac{\hbar^2}{4}$$

(2) Time Development of the Oscillator

– Heisenberg picture "

EOM : $\dfrac{d\tilde{p}(t)}{dt} = \dfrac{1}{i\hbar}\left[\tilde{p}(t), H\right]$

$p(t) \equiv p^{(H)}(t)$
$x(t) \equiv x^{(H)}(t)$

$\dfrac{d\tilde{x}(t)}{dt} = \dfrac{1}{i\hbar}\left[\tilde{x}(t), H\right]$

NOTE: We know that $\langle H\rangle$ is conserved, thus, $H(t) = H$ (t-indep.)
But, let's just try with $H(t)$.

$H = \dfrac{\tilde{p}(t)^2}{2m} + \dfrac{1}{2}m\omega^2\tilde{x}(t)^2$

* show $[x(t), p(t)] = i\hbar$
when $x(t) = e^{\frac{iH}{\hbar}t}\, x\, e^{-\frac{iH}{\hbar}t}$
$p(t) = e^{\frac{iH}{\hbar}t}\, p\, e^{-\frac{iH}{\hbar}t}$

$$\Rightarrow \left[\begin{array}{l} \dfrac{d\tilde{p}(t)}{dt} = -m\omega^2\tilde{x}(t) \\[2ex] \dfrac{d\tilde{x}(t)}{dt} = \dfrac{p(t)}{m} \end{array}\right.$$

$$\Rightarrow \frac{d}{dt}\begin{pmatrix}\tilde{x}(t)\\ \tilde{p}(t)\end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{m} \\ -m\omega^2 & 0\end{pmatrix}\begin{pmatrix}\tilde{x}(t)\\ \tilde{p}(t)\end{pmatrix}.$$

$\equiv A$

diagonalization :

eigenvalues $\left[\begin{array}{l} i\omega \leftarrow \begin{pmatrix}-\frac{i}{m\omega}\\ 1\end{pmatrix} \\ -i\omega \leftarrow \begin{pmatrix}1\\ -im\omega\end{pmatrix}\end{array}\right.$

$\hookrightarrow$

$$\underline{X^{-1}AX = D}$$

$$D = \begin{pmatrix} i\omega & 0 \\ 0 & -i\omega\end{pmatrix}$$

$\hookrightarrow$ $\underline{A = XDX^{-1}}$

$$X = \frac{1}{\sqrt{2}}\begin{pmatrix} -\frac{i}{m\omega} & 1 \\ 1 & -im\omega\end{pmatrix}$$

$$\Rightarrow \frac{d}{dt}\left[X^{-1}\begin{pmatrix}\tilde{x}(t)\\ \tilde{p}(t)\end{pmatrix}\right] = D\left[X^{-1}\begin{pmatrix}\tilde{x}(t)\\ \tilde{p}(t)\end{pmatrix}\right].$$

$$X^{-1}\begin{pmatrix}\tilde{x}(t)\\ \tilde{p}(t)\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} i m\omega & 1 \\ 1 & \frac{i}{m\omega}\end{pmatrix}\begin{pmatrix}\tilde{x}(t)\\ \tilde{p}(t)\end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} i m\omega\,\tilde{x}(t) + \tilde{p}(t) \\ \tilde{x}(t) + \frac{i}{m\omega}\tilde{p}(t)\end{pmatrix} \equiv A\begin{pmatrix}\hat{\varphi}_1(t)\\ \hat{\varphi}_2(t)\end{pmatrix}$$

→ Any diagonal matrix with c-numbers. ∴ Choose it to be "I".

$$\Rightarrow \frac{d}{dt}\begin{pmatrix}\hat{\varphi}_1(t)\\ \hat{\varphi}_2(t)\end{pmatrix} = \begin{pmatrix} i\omega & 0 \\ 0 & -i\omega\end{pmatrix}\begin{pmatrix}\hat{\varphi}_1(t)\\ \hat{\varphi}_2(t)\end{pmatrix}.$$

$$\therefore \boxed{\begin{aligned}\hat{\varphi}_1(t) &= c_1 e^{i\omega t} \\ \hat{\varphi}_2(t) &= c_2 e^{-i\omega t}\end{aligned}} \qquad \begin{aligned} c_1 &= \hat{\varphi}_1(0) \\ c_2 &= \hat{\varphi}_2(0)\end{aligned}$$

also for $\tilde{a}^\dagger$, $\tilde{a}$

time-invariant : $\varphi_1(t)\cdot\varphi_2(t) = c_1 c_2$.

$$\left[\tilde{x}(t) - \frac{i}{m\omega}\tilde{p}(t)\right] = \left[\tilde{x}(0) - \frac{i}{m\omega}\tilde{p}(0)\right] e^{i\omega t}$$

$$\left[\tilde{x}(t) + \frac{i}{m\omega}\tilde{p}(t)\right] = \left[\tilde{x}(0) + \frac{i}{m\omega}\tilde{p}(0)\right] e^{-i\omega t}$$

Q. Is it really classical?

$$\Rightarrow \left[\begin{aligned}\tilde{x}(t) &= \tilde{x}(0)\cos\omega t + \frac{\tilde{p}(0)}{m\omega}\sin\omega t \\ \tilde{p}(t) &= -m\omega\,\tilde{x}(0)\sin\omega t + \tilde{p}(0)\cos\omega t\end{aligned}\right.$$

$$\cos\omega t = \frac{e^{i\omega t} + e^{-i\omega t}}{2}, \qquad \sin\omega t = \frac{e^{i\omega t} - e^{-i\omega t}}{2i}$$

↑ These can be verified by using the Baker-Hausdorff Lemma : $A(t) = e^{\frac{iHt}{\hbar}} A e^{-\frac{iHt}{\hbar}}$

$= \cdots$ (pp 95) of Sakurai.

☆ What about $\tilde{a}$, $\tilde{a}^\dagger$?

Note that, $\hat{\varphi}_1 = \frac{1}{\sqrt{2}}(i m\omega\,\tilde{x} + \tilde{p})$

$\propto \tilde{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\tilde{x} - \frac{i\tilde{p}}{m\omega}\right)$.

and $\hat{\varphi}_2 \propto \tilde{a}$; $\Rightarrow \boxed{\begin{aligned}\tilde{a}^\dagger(t) &= e^{i\omega t}\,\tilde{a}^\dagger(0) \\ \tilde{a}(t) &= e^{-i\omega t}\,\tilde{a}(0)\end{aligned}}$

∴ $\tilde{a}^\dagger(t)\,\tilde{a}(t)$ = time-invariant. ; $[H, \tilde{a}^\dagger\tilde{a}] = 0$.

→ simultaneous eigenket !!

Look at :

$$\begin{cases} \tilde{x}(t) = \tilde{x}(0)\cos\omega t + \dfrac{\tilde{p}(0)}{m\omega}\sin\omega t \\ \tilde{p}(t) = -m\omega\,\tilde{x}(0)\sin\omega t + \tilde{p}(0)\cos\omega t \end{cases}.$$

$$\Leftarrow \quad m\frac{d\tilde{x}}{dt} = \tilde{p}, \quad \text{just like CM.}$$

But, $\langle\tilde{x}\rangle$ and $\langle\tilde{p}\rangle$ are "not" oscillating. !!!

although $\tilde{x}(t)$ and $\tilde{p}(t)$ look like oscillating.

|| NOTE : $\langle\tilde{x}\rangle = 0$, $\langle\tilde{p}\rangle = 0$, for all $|n\rangle$.

Q. Can we find a "Quantum" state

that behaves just like classical $\langle x\rangle$ and $\langle p\rangle$?

* Coherent States . ← This is the one.

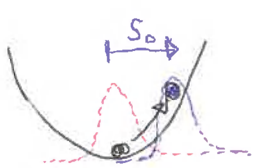
why do we need this ?

- We live in a "classical" world,

But we want to control a "Quantum" world.

: We need a "bridge" !

the easiest way to make the coherent state

→  $|s_0\rangle = J(s_0)\,|0\rangle$

move the ground state to s_0.

wave function $\psi_{s_0}(x) = \psi_0(x - s_0)$. || $\langle x|J(s_0) = \langle x - s_0|$

observables :

$\langle s_0|\tilde{x}|s_0\rangle = \langle 0|J^\dagger(s_0)\,\tilde{x}\,J(s_0)|0\rangle = s_0$

$\langle s_0|\tilde{p}|s_0\rangle = 0$.

$\langle s_0|H|s_0\rangle = \langle 0|\frac{\tilde{p}^2}{2m}|0\rangle + \frac{1}{2}m\omega^2\langle 0|(\tilde{x}+s_0)^2|0\rangle$.